Information theoretic approach to quantify complete and phase synchronization of chaos

A. Shabunin, V. Demidov, V. Astakhov, and V. Anishchenko
Department of Physics, Saratov State University, Astrakhanskaya 83, Saratov, Russia

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Based on an information theory approach we suggest a quantitative characteristic for evaluating the degree of chaotic synchronization. The proposed characteristic is tested for the cases of complete and phase synchronization of chaos. It is shown that this characteristic is stable with respect to the influence of small noise and nonlinear signal distortion.

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I. INTRODUCTION

Recently investigation of interacting complex systems has been the focus of attention of much research. The interacting behavior that is often called ''chaotic synchronization'' refers to a number of different phenomena such as a transition to completely identical oscillations in coupled subsystems (complete synchronization of chaos [1,2]), basic frequency locking (frequency synchronization [3]), instantaneous phase locking (phase synchronization [4]), a deterministic relationship between the dynamics of oscillators (lag synchronization [5] and generalized synchronization [6]). For each phenomenon listed, one can separate cases of full and partial synchronization. To gain a better understanding of the correlation between different types of chaotic synchronization it would be useful to introduce a quantity for measuring the degree of interdependence between the motions of subsystems. In our opinion, this quantity must satisfy the following necessary requirements.

(1) It must be universal in order to be applied to different types of behavior of interacting oscillators. It has to give the possibility to compare different stages of the particular type of synchronization as well as different types of synchronization with each other.

(2) It must represent a normalized quantity from 0 for unsynchronized oscillations to 1 for fully synchronized ones.

(3) It must have a clear physical meaning that can facilitate the interpretation of obtained results.

(4) This quantity must be independent of a particular type of the dynamical system, thus, allowing us to determine the degree of synchronization by using the time series of oscillations in the subsystems.

In Ref. [7] we suggested a synchronization measure, that was introduced on the basis of the coherence function, and tested it on an example of complete synchronization loss in a system of two coupled chaotic self-sustained oscillators. The phase coherence was used as a chaotic synchronization measure in Ref. [8], where the authors applied the instantaneous phase approach.

In this work we consider another approach for constructing the chaotic synchronization measure and use the information quantity function. We apply this approach to the case of unidirectional coupling between oscillators when one of them can be considered as a transmitter of a chaotic signal and the other one as a receiver. In the framework of this approach we assume that the degree of synchronization can be represented by the amount of information provided by knowledge of the state of the transmitter to determine the state of the receiver. If the state of one oscillator uniquely determines the state of another, we can speak on full synchronization of chaos. Hence, when states of interacting subsystems are connected through a deterministic function (similarly to the definition of generalized synchronization), the level of synchronization is equal to 1. On the other hand, if the state of the transmitter does not influence the state of the receiver, we can conclude that the oscillators are unsynchronized (zero degree of synchronization). In certain cases the correlation between states of the oscillators has two components, deterministic and random. This refers to partial synchronization of chaos when the degree of synchronization is between 0 and 1.

We introduce the following quantitative characteristic of synchronization:

$$\mu = \frac{S_y - S_{y|x}}{S_y}, \tag{1}$$

where S_y is the information entropy that is calculated on the distribution of states of the synchronized oscillator, $S_{y|x}$ is the conditional entropy computed when the state of the synchronizing oscillator is fixed at a certain value of x. This approach is based on methods of symbolic analysis [9–11]. A series of works have demonstrated that the methods of symbolic dynamics can be used to reveal a similarity of complex signals. In Ref. [10] the conditional entropy built on a symbolic sequence was applied to identify chaotic signals. In Ref. [11] the authors use the mutual information to observe synchronization in unidirectionally coupled Lorenz and Rossler systems and to study the electrical activity in human brains.

In this work we apply the chaos synchronization to measure two qualitatively different types of chaotic synchronization. In the first case we consider the process of destruction of complete chaotic synchronization in a system of two unidirectionally coupled logistic maps. The second case is concerned with the phase locking process in the Rossler system driven by an external harmonic force.

II. ESTIMATION OF THE DEGREE OF CHAOTIC SYNCHRONIZATION IN A SYSTEM OF COUPLED LOGISTIC MAPS

In this section we consider the case of breaking of complete synchronization of chaos in a system of maps with master-slave coupling,

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$$x_{n+1}=\lambda x_n(1-x_n), \tag{2}$$

$$y_{n+1}=\lambda[y_n+\gamma(x_n-y_n)]\{1-[y_n+\gamma(x_n-y_n)]\}, \tag{3}$$

where x_n, y_n are dynamical variables, λ is the system parameter, and γ is the coupling strength. If $\gamma=0$, the oscillators are uncoupled. Equations (2) and (3) describe the drive and response systems, respectively. This type of coupling has been considered in detail in Ref. [12]. In a certain interval of the coupling parameter, the system exhibits a phenomenon of complete synchronization of chaos, which manifests itself in identical oscillations of the subsystems, i.e., $x_n=y_n$. In the region of chaos, for each value of λ there is a critical value of the coupling below which complete synchronization is broken. The breaking of synchronization is accompanied by a bubbling phenomenon followed by the blowout bifurcation [13], which is resulted in asynchronous chaotic oscillations. The system dynamics and mechanisms of synchronization in it were described in Ref. [12]. We study the behavior of the system (2) and (3) depending on the coupling parameter γ and for fixed $\lambda=3.8$. The chosen value of λ corresponds to the regime of developed chaos represented by a one-band chaotic attractor. In the region $0.38<\gamma<1$ we observe a robust regime of complete synchronization. As the coupling parameter decreases, $0.35<\gamma<0.38$, the synchronization of chaos is no longer robust. Synchronous oscillations can be observed after a long transient process only in the system without noise. The addition of an arbitrary small noise leads to the bubbling behavior. The time series x-y represents long intervals of the synchronous behavior that are intermitted by random turbulent bursts, when the trajectory goes away from the symmetric subspace. This behavior is illustrated in Figs. 1(a) and 1(b) where the phase portraits of the attractor are shown both without noise [Fig. 1(a)] and in the presence of small noise [Fig. 1(b)]. As the coupling parameter decreases further, the transversal Lyapunov exponent becomes positive at $\gamma=0.35$. This leads to the blowout bifurcation after which the chaotic synchronization is no longer observed in the system. With decreasing coupling the attractor "inflates" and at $\gamma<0.05$ the phase portrait looks like a square region [Fig. 1(c)].

We use the proposed characteristic μ for evaluating the destruction of complete synchronization. Figure 2 shows a dependence of the degree of synchronization on the coupling in the system (2) and (3) without noise and in the presence of small noise added. In the noiseless system, the sharp decay of the synchronization degree begins immediately at the point where the transversal Lyapunov exponent changes its sign. For illustration we plot the transversal Lyapunov exponent as a function of the coupling parameter and two dotted lines (horizontal and vertical) the crossing of which indicates the point where the exponent passes through its zero value. It is seen that the degree of synchronization sharply drops from $\mu=1$ at $\gamma\geqslant 0.35$ to $\mu\simeq 0.15$ at $\gamma=0.3$. Then, as parameter γ decreases further, the degree of synchronization slowly decreases to $\mu\simeq 0.05$ at $\gamma=0.2$. This region corresponds to the bubbling phenomenon. The local increase of μ at $\gamma\sim 0.15$ is related to a more regular structure of the attractor represented in Fig. 1(d). After this the degree of synchronization tends to zero at zero coupling. In the presence of small noise ($\sim 0.000\,01$) a rapid decrease in the synchronization degree begins significantly earlier (with respect to the coupling) as indicated by dot-dashed curve in Fig. 2. This process goes more gradually than that in the noiseless system. A significant difference in the degrees of synchronization for the system with and without noise is observed only in this region up

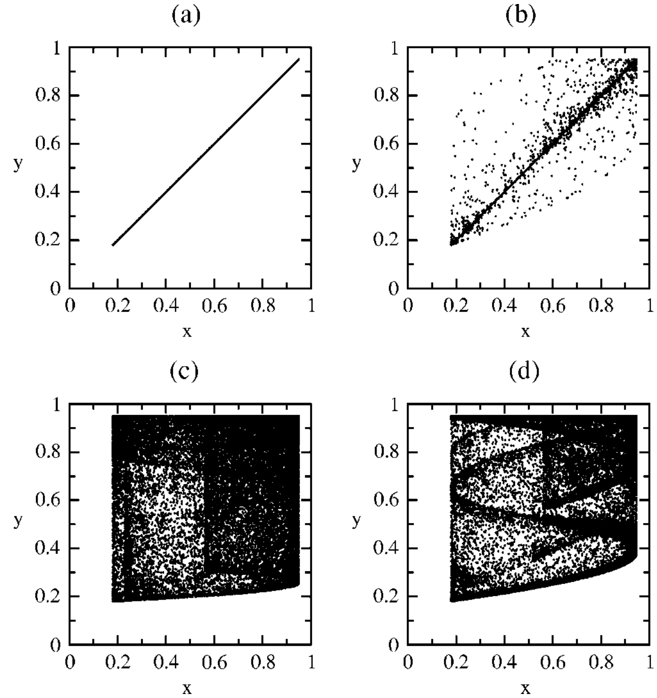

FIG. 1. Breaking of complete chaotic synchronization in the system (2) and (3) for different coupling strength: bubbling phenomenon for $\gamma=0.36$ without noise (a) and in the presence of weak noise (intensity $\sim 0.000\,01$) (b); unsynchronous chaos for $\gamma=0.05$ (c) and partial regularization of oscillations at $\gamma=0.14$ (d).

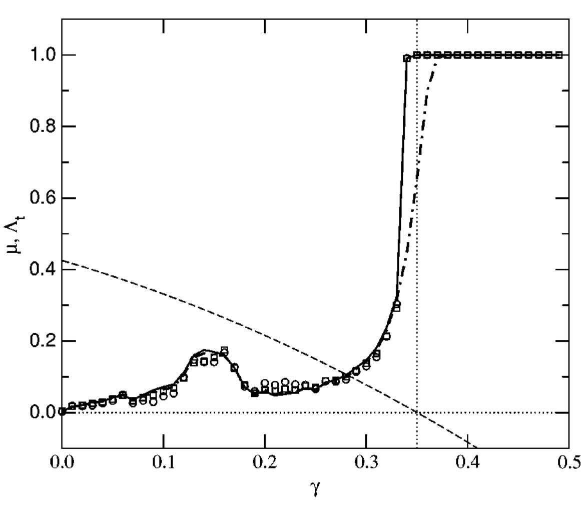

FIG. 2. Dependences of the degree of chaotic synchronization μ (dash-dotted and solid curves for noisy and noiseless systems, respectively, and symbols "□" and "○" for the system with signal distortions) and the transversal Lyapunov exponent on the coupling for the master-slave system.

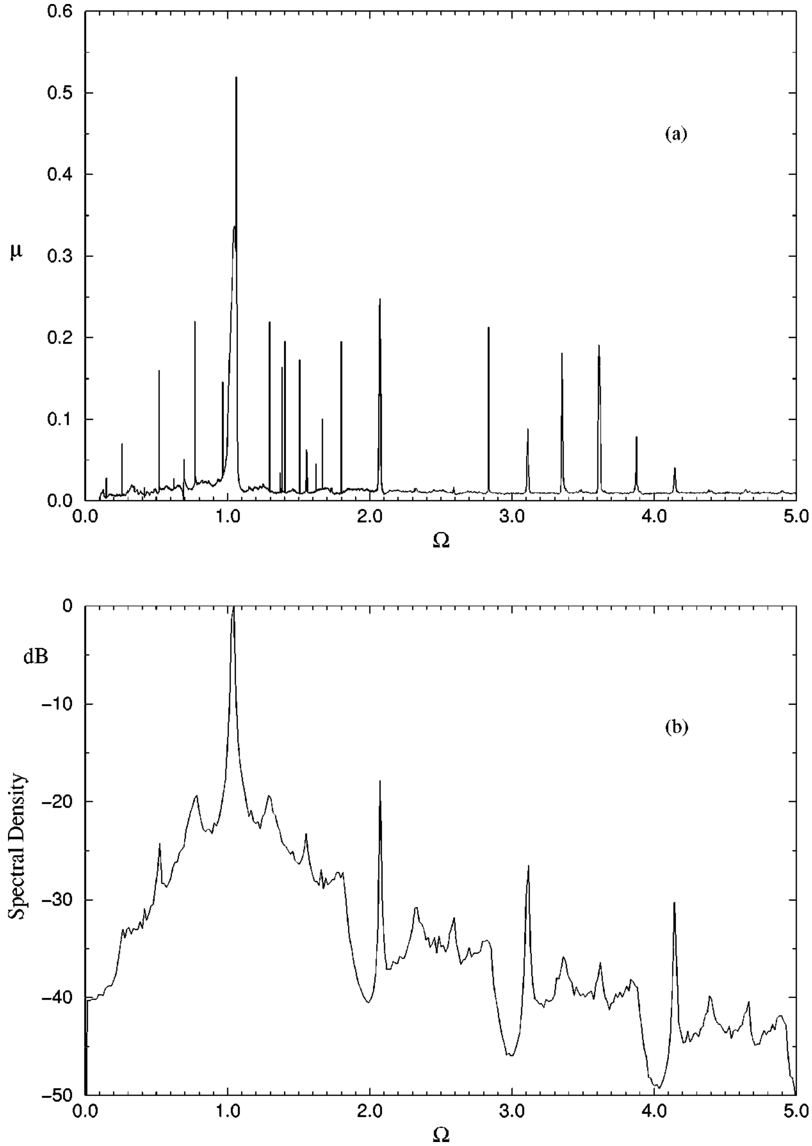

FIG. 3. (a) Dependence of the degree of phase chaotic synchronization μ on the external force frequency Ω at $A=0.5$, and (b) the power spectrum of the autonomous Rossler system.

to $\gamma \simeq 0.34$. This difference indicates the bubbling process caused by external noise. In other regions of the coupling parameter the values of μ are practically the same in both cases (Fig. 2). Thus, the degree of synchronization is significantly affected by noise only in regions where chaos synchronization is nonrobust.

The proposed characteristic of synchronization is stable not only to a weak noise but to a small nonlinear distortion of signals. To illustrate this statement, we modify the time series that is used to calculate the degree of synchronization, by adding a nonlinear term

$$y \rightarrow y + \delta y^2,$$

where δ is a small parameter. The results of calculations for the modified realizations with $\delta=0.05$ and $\delta=0.2$ are shown in Fig. 2 (they are indicated by circles and squares, respectively). As it is seen, the degree of synchronization found on the changed signal does not practically differ from the initial undistorted case. The invariance with respect to nonlinear distortions enables us to apply the proposed characteristic to the study of generalized synchronization of chaos when $y_n = f(x_n)$ with f being a deterministic function.

Our investigations have demonstrated that the proposed measure of quantity of synchronization is sensitive to breaking of complete chaotic synchronization. The degree of synchronization is exactly equal to 1 in the case of totally synchronized oscillators and it is close to 0 when oscillations are unsynchronous at a very small coupling. This quantity is stable with respect to the influence of weak noise (except the region with bubbling behavior) and nonlinear distortion of the signal being studied.

III. EVALUATING THE DEGREE OF PHASE SYNCHRONIZATION OF CHAOS

The phenomenon of phase synchronization is another type of chaotic synchronization that attracts the interest of many researchers and has a great fundamental and applied significance. We test our method on a well-known system that demonstrates phase synchronization of chaos: the Rossler oscil-

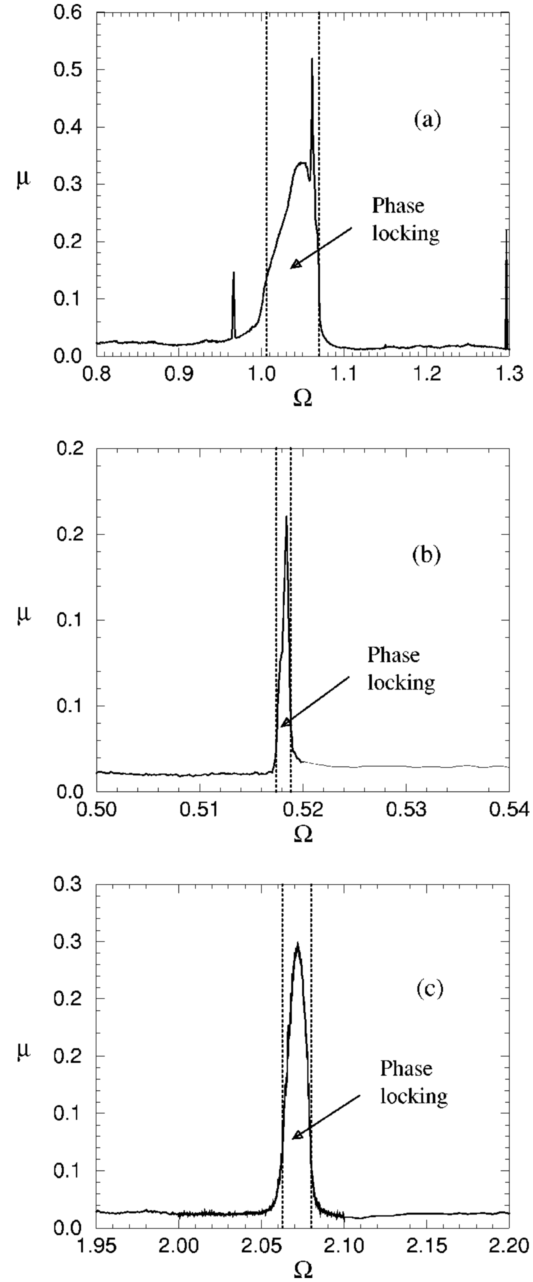

FIG. 4. Dependence of the degree of phase chaotic synchronization μ on Ω for the Rossler system (a) $1.0060<\Omega<1.0689$ (main resonance $\Omega\approx\omega_0$), (b) $0.5174<\Omega<0.5188$ (resonance $2\Omega\approx\omega_0$), (c) $2.0625<\Omega<2.0801$ (resonance $\Omega\approx2\omega_0$).

lator with an external harmonic force:

$$\dot{x}=-y-z+A\cos\Omega t,$$

$$\dot{y}=x+ay, \tag{4}$$

$$\dot{z}=b+z(x-c).$$

Here x,y,z are dynamical variables, a, b, and c are controlling parameters, A and Ω are the amplitude and the frequency of the external force, respectively. We have chosen a chaotic regime and fixed the parameters as follows $a=0.15, b=0.15, c=10$. For certain values of the force amplitude and frequency the system (4) demonstrates the phenomenon of phase synchronization between the chaotic dynamics of the oscillator and the periodic external force [14–16].

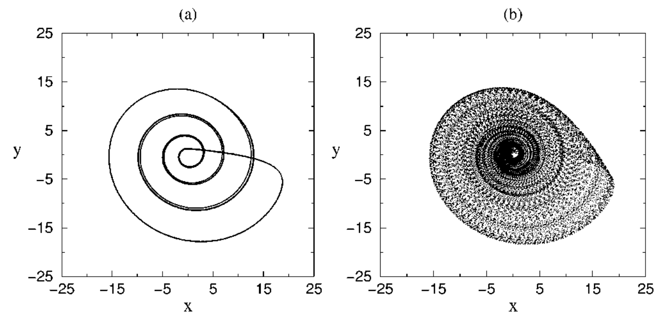

FIG. 5. Phase portraits of the system (4): (a) $\Omega=1.061$, $A=0.5$, (b) $\Omega=1.036$, $A=0.5$.

The phase synchronization is defined as locking of the instantaneous phase of oscillations of the synchronized system by the phase of the external force. The condition for full phase synchronization on the basic frequency reads

$$|\phi_1-\phi_2|\leqslant\text{const}. \tag{5}$$

In this case the basic frequency in the spectrum of the studied signal and the frequency of the external force are close to each other. In the case of synchronization on subharmonics this condition is as follows:

$$|n\phi_1-m\phi_2|\leqslant\text{const}. \tag{6}$$

In expressions (5) and (6) ϕ_1 and ϕ_2 are the instantaneous phases, and n and m are integer numbers. In the case of partial phase synchronization both conditions (5) and (6) are no longer valid and the phase difference of the phases tends to infinity with time. Nevertheless, in this case the phase locking exists during very long time intervals between which rapid skips of the phase difference are observed.

We investigate the system (4) in a wide region of the external force frequency for $A=0.5$. Figure 3(a) demonstrates the dependence of the synchronization degree μ on

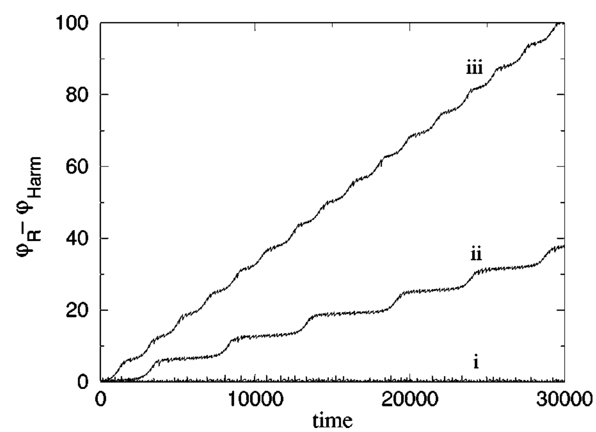

FIG. 6. Temporal dependences of the phase difference ($\varphi_R-\varphi_{Harm}$) for $A=0.5$, and (i) $\Omega=1.0264$, (ii) $\Omega=1.071$, (iii) $\Omega=1.082$.

nance character with several peaks. The maximal peak corresponds to the main resonance, which is located on the basic frequency $\omega_0 = 1.061$ in the spectrum of the Rossler oscillator [Fig. 3(b)]. This peak indicates the region of phase synchronization $(1:1)$ where $n = m = 1$ [see Eq. (6)]. Figure 4(a) shows this region in a larger scale. The frequency interval corresponding to the full phase synchronization $(1.0061 < \Omega < 1.0689)$ is bounded by the dotted lines. The largest value of the degree of synchronization is $\mu = 0.519$ while the minimal value is about 0.15. The value of the synchronization degree depends on an oscillatory regime in the system (4) and it is significantly larger for more regular oscillations in the windows of periodicity. For example, the sharp peak of μ at frequency $\Omega = 1.061$ refers to the period -16 limit cycle [Fig. 5(a)]. This peak is observed near the local maximum $\mu = 0.34$ at the basic frequency $\Omega = \omega_0$. The other values of μ in this region belong to the interval $[0.15;0.34]$ and are related to chaotic synchronous oscillations [Fig. 5(b)].

Depending on the frequency of the external force the oscillations in the system can be totally synchronized [curve (i) in Fig. 6], partially synchronized [curve (ii)], or unsynchronized [curve (iii)]. Starting from the region of partial phase synchronization we observe that when the system parameter changes in one direction, the time intervals of phase locking are gradually increased to infinity, while they are gradually decreased to zero as the parameter is varied in the opposite direction. This leads to a gradual change of the degree of synchronization from the values in the interval $0.15 < \mu < 0.34$ for totally synchronized oscillations to $0.01 < \mu < 0.15$ for partially synchronized and then to almost zero value for unsynchronized ones.

The other peaks presented in Fig. 3(a) reflect synchronization on harmonics. For example, Fig. 4(b) represents a peak of synchronization at $1:2$ when the condition $|\phi_1 - 2\phi_2| \leqslant \text{const}$ is fulfilled. Figure 4(c) relates to the case of synchronization at $2:1$. Each peak in Fig. 3 corresponds to phase synchronization with a certain rational number. Higher resonances have smaller synchronization levels and narrower regions of synchronization than the main resonance.

Comparing complete and phase synchronization of chaos we find that the first case is characterized by a significantly larger degree of synchronization. This is obvious since complete synchronization means correlations between amplitudes and phases of signals whereas phase synchronization takes into account only phase dynamics. Hence, full phase synchronization is characterized by the synchronization degree being significantly less than 1. From this point of view, we can consider phase synchronization as an example of partially synchronous oscillations.

IV. CONCLUSION

We have proposed the characteristic of chaotic synchronization that can be applied to different types of synchronous behavior. We have tested it on examples of complete and phase synchronization of chaos. In the first case the degree of synchronization is exactly equal to 1 for totally synchronized oscillations and is reduced when synchronization is broken. In the bubbling region the degrees of synchronization without and in the presence of small noise are very different. This behavior reflects a nonrobust (or ''weak'') character of synchronization. In the other parameter regions weak noise does not practically influence on the degree of synchronization. We have demonstrated that its value is also invariant to a small distortion of the signal.

In the case of phase synchronization the suggested measure indicates regions of synchronization with different rational numbers. The degree of synchronization is significantly less than that for complete synchronization. It depends on which kind of an oscillatory regime is observed in the oscillator. Higher resonances have smaller levels of synchronization than the main one.

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